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THE *BP* HOMOLOGY OF *H*-SPACES[1]

BY

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ABSTRACT. The *BP* homology of 1-connected *H*-spaces whose loop space is torsion free is studied. It is shown that localizing in a suitable manner kills all torsion in $BP_*(X)$. Similar results are then obtained for the bordism of X. Finally the *K*-theory of X is shown to have no p torsion.

1. Introduction. By an H-space we will mean a pointed topological space X which has the homotopy type of a connected CW complex of finite type together with a basepoint preserving map μ: $X \times X \to X$ with two sided homotopy unit. In this paper we will study, for each prime p, the Brown-Peterson homology, $BP_*(X)$, of X when (X, μ) is a 1-connected H-space and $H_*(\Omega X)$ has no p torsion. Here ΩX is the loop space of X and $H_*(\ )$ denotes ordinary homology with Q_p coefficients (Q_P are the integers localized at the prime p). We will proceed by using this assumption on ΩX to study $BP_*(\Omega X)$ and then use a spectral sequence to reinterpret the results in terms of $BP_*(X)$. This type of approach is not new. It has often been used in ordinary homology (see **[6]**, **[9]** or **[16]**). Petrie demonstrated in **[24]** that it could be used in bordism as well, at least for compact Lie groups. This paper is motivated by **[24]** and uses many of its ideas. However, we will first state our theorems before explaining just how our results and arguments are related to Petrie's.

The Brown-Peterson homology $BP_*(X)$ is a module over $\Lambda = BP_*(\mathrm{pt}) = Q_p[v_1, v_2, \dots]$ ($\deg v_s = 2p^s - 2$). Thus, besides p torsion, we can also talk about v_s torsion for $s \geqslant 1$. These torison submodules are interrelated. In particular, letting $v_0 = p$ and, for each $s \geqslant 0$, letting M_s be the v_s torsion submodule, we have the inclusions $\dots \subset M_{s+1} \subset M_s \subset \cdots \subset M_0$ (see 3.2 of **[15]**. Our work is independent of the results of **[15]**. However, the work of Johnson and Wilson certainly provides a useful perspective on this paper). The bulk of this paper is devoted to showing that the torsion of $BP_*(X)$ is all contained in M_1. We do this by showing that $BP_*(X)$ localized with respect

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to v_1 is torsion free. Let $\Lambda(1) = \Lambda(1/v_1)$ and $BP_*(X;\ \Lambda(1)) = BP_*(X) \otimes_\Lambda \Lambda(1)$.

THEOREM 1.1. *$BP_*(X;\ \Lambda(1))$ is torsion free.*

In the process of proving 1.1 we also show that $BP_*(X;\ \Lambda(1))$ agrees, as a $\Lambda(1)$ module, with $\mathrm{Tor}^{BP_*(\Omega X;\Lambda(1))}(\Lambda(1);\ \Lambda(1))$. For we prove 1.1 by studying the algebra structure of $BP_*(\Omega X;\ \Lambda(1))$ and then passing to $BP_*(X;\ \Lambda(1))$ via an Eilenberg-Moore type spectral sequence. It might be expected, in view of **[15]**, that 1.1 can be strengthened to assert that $BP_*(X;\ \Lambda(1))$ is $\Lambda(1)$ free. However, such is not the case. This generalization will be discussed in a sequel.

We can also deduce results about the algebra structure of $BP_*(X;\ \Lambda(1))$. Let Ω_*: $Q(BP_*(\Omega X;\ \Lambda(1))) \to P(BP_*(X;\ \Lambda(1)))$ be the delooping map (see §3).

THEOREM 1.2.(a) *$BP_*(X;\ \Lambda(1))$ is a commutative (associative) if, and only if, $H_*(X;\ Q)$ is commutative (associative).*

(b) *If $H^*(X;\ Q)$ is an exterior algebra then $BP_*(X;\ \Lambda(1))$ is generated as an algebra by* Image Ω_*.

These results about BP homology can be used to deduce results about complex bordism and K-theory. Recall that the bordism ring $\Omega = MU_*(\mathrm{pt}) \otimes Q_p$ is a free Λ module via a canonical inclusion (see **[25]**). This enables us to define the ring $\Omega(1) = \Omega(1/v_1)$ and the homology theory $MU_*(X;\ \Omega(1)) = MU_*(X) \otimes_\Omega \Omega(1)$.

THEOREM 1.3. *Theorems* 1.1 *and* 1.2 *are true when $BP_*(X;\ \Lambda(1))$ is replaced by $MU_*(X;\ \Omega(1))$.*

Using 1.3 and the Conner-Floyd relation between bordism and K-theory (see **[11]** or **[24]**) we can then deduce the most striking of our results.

THEOREM 1.4. *Theorems* 1.1 *and* 1.2 *are true when $BP_*(X;\ \Lambda(1))$ is replaced by $K_*(X) \otimes_Z Q_p$.*

As we mentioned above, our work is motivated by that of Petrie in **[24]**. In that paper he established results similar to 1.3 and 1.4 when X is a 1-connected semisimple compact Lie group (prime to the exceptional groups E_7 and E_8 when $p = 2$). His method was to study the algebra structure of $MU_*(\Omega X;\ G)$ and then pass to $MU_*(X;\ G)$. However, his results depended on the explicit geometry of the Lie groups involved. For he used Bott's structure theorems from [3] as well as the theory of characteristic classes. In this paper we work purely from the homotopy theoretic point of view. We replace Bott's results by our assumption on ΩX. We then work in BP theory rather than in MU

theory. First, $BP_*(X)$ is a direct summand of $MU_*(X) \otimes Q_p$ which generates $MU_*(X) \otimes Q_p$ as an Ω module. Thus no information about $MU_*(X) \otimes Q_p$ is lost. More importantly, we are then able to use the operations from $BP^*(BP)$ to study the algebra structure of $BP_*(\Omega X;\ G)$. The use of these operations constitutes the main innovation of this paper. They replace the use of characteristic classes of Petrie. We use these operations to show that the algebra structure of $BP_*(\Omega X)$ reflects not merely the algebra structure of $H_*(\Omega X;\ \mathbf{Z}/p)$ but also its structure as a Hopf algebra over the Steenrod algebra.

Our theorems are also related to other results from finite H-space theory. The hypothesis that $H_*(\Omega X)$ is torsion free is conjectured to be true whenever (X, μ) is a 1-connected mod p finite H-space. It has been established for p odd (see [21]). That the K-theory of X, $K_*(X) \otimes Q_p$, is torsion free is also a standard conjecture for 1-connected mod p finite H-spaces. Again this is known to be true when p is odd (see [21]). Thus our arguments tie these results together and, furthermore, show that the relations do not depend on the finiteness of the spaces involved. The fact that proving $K_*(X) \otimes Q_2$ is torsion free reduces to showing that $H_*(\Omega X)$ is torsion free (Q_2 coefficients) may be important when X is mod 2 finite. For there are techniques to show that $H_*(\Omega X)$ is torsion free (see [16]) while the approach used in [21] to show $K_*(X) \otimes Q_p$ is torsion free for p odd runs into difficulties when $p = 2$. We will discuss the relation of our method to that of Lin's in a sequel to this paper.

The organization of this paper is as follows. In §2 we discuss BP homology and cohomology. In §3 we discuss an Eilenberg-Moore type spectral sequence which relates $BP_*(\Omega X;\ G)$ to $BP_*(X;\ G)$. In §4 we discuss the structure of $H_*(\Omega X;\ \mathbf{Z}/p)$ as a Hopf algebra over the Steenrod algebra. These results are then used in §§5 and 6 to discuss the structure of $BP_*(\Omega X)$. In §7 we prove Theorem 1.1 by using the results of §§5 and 6 plus the spectral sequence from §3. In §7 we prove Theorems 1.2–1.4.

We will work with the following conventions. All spaces are assumed to have the homotopy type of a connected CW complex of finite type. All spaces have basepoints and all maps preserve basepoints. The basic ideas of generalized homology and cohomology theory (see [2]) will be assumed. The basics of Hopf algebra theory (see [23]) will also be assumed. We will use the symbols P and Q in a functorial manner to denote primitives and indecomposables, respectively. Given a set X we will use the symbols $E(X)$, $\Gamma(X)$, and $F(X)$ to denote the exterior, divided polynomial, and polynomial algebras, generated by the set X over the ring F. All tensor products are over $\mathbf{Z}$ unless otherwise indicated. We will be using the same symbol to denote an element and its image in a quotient module.

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2. *BP* homology and cohomology. In this section we discuss BP homology and cohomology. For the purposes of this section X will be an arbitrary connected CW complex. For any such X, $BP_*(X)$, the BP homology of X, is a canonical direct summand of $MU_*(X) \otimes Q_p$ and $BP^*(X)$, the BP cohomology of X, is a canonical direct summand of $MU^*(X) \otimes Q_p$ (see [25]). We can regard $BP^*(X)$ as a Λ module via the identities

$$BP^i(pt) = BP_{-i}(pt). \tag{2.1}$$

Thus Λ will be considered as negatively graded when we deal with cohomology.

Coefficients other than $\Lambda = BP_*$ can be introduced into BP homology and cohomology. We will use the coefficients

$$\Lambda_p = \Lambda \otimes \mathbf{Z}/p = \mathbf{Z}/p[v_1, v_2, \ldots], \qquad \Lambda_p(1) = \Lambda_p(1/v_1),$$
$$\Lambda_0 = \Lambda \otimes Q = Q[v_1, v_2, \ldots], \qquad \Lambda_0(1) = \Lambda_0(1/v_1).$$

These coefficients are introduced into BP theory by modifying the BP spectrum (see [2]). However, the various theories can then be shown to be interrelated in the following manner:

$$\begin{aligned}
&\text{(a) } BP_*(X; \Lambda(1)) = BP_*(X) \otimes_\Lambda \Lambda(1),\\
&\text{(b) } BP_*(X; \Lambda_p(1)) = BP_*(X; \Lambda_p) \otimes_{\Lambda_p} \Lambda_p(1),\\
&\text{(c) } BP_*(X; \Lambda_0(1)) = BP_*(X; \Lambda_0) \otimes_{\Lambda_0} \Lambda_0(1),\\
&\text{(d) } BP_*(X; \Lambda_0) = BP_*(X) \otimes Q,\\
&\text{(e) } BP_*(X; \Lambda_0(1)) = BP_*(X; \Lambda(1)) \otimes Q,\\
&\text{(f) } BP_*(X; \Lambda_0) = H_*(X; Q) \otimes_Q \Lambda_0
\end{aligned} \tag{2.2}$$

(for (a), (b), (c), at least in the analogous case of $MU_*(X)$, see §2 of [24]; (d) and (e) follow by similar arguments; for (f) see [19]). Thus we could use 2.2 to define all the theories except $BP_*(X; \Lambda_p)$ (and it is just $BP_*(X)$ with mod p coefficients).

Let $H_*(X)$ and $H^*(X)$ denote ordinary homology and cohomology with Q_p coefficients. The BP homology and cohomology are related to ordinary homology and cohomology by the following commutative diagrams:

$$\begin{array}{ccccccc}
BP_*(X) & \xrightarrow{T} & H_*(X) & \quad & BP^*(X) & \xrightarrow{T} & H^*(X)\\
\downarrow\rho & & \downarrow\rho & & \downarrow\rho & & \downarrow\rho\\
BP_*(X; \Lambda_p) & \xrightarrow{T} & H_*(X; Z/p) & & BP^*(X; \Lambda_p) & \xrightarrow{T} & H^*(X; Z/p)
\end{array} \tag{2.3}$$

The maps are all induced by canonical maps between the spectra representing the various theories. The vertical maps are reduction mod p while the horizontal maps are the Thom maps (see [26]). We use the symbols T and ρ for more than one map since it will be clear from the context which specific map is meant. The maps respect the pairings between homology and cohomology for the various theories. That is, we have identities of the form $T\langle x, y\rangle = \langle T(x), T(y)\rangle$ and $\rho\langle x, y\rangle = \langle\rho(x), \rho(y)\rangle$.

We define filtrations $\{F_q\}$ on $BP_*(X; G)$ and $BP^*(X; G)$ for $G = \Lambda$ or Λ_p as follows. Filter Λ by the rule $F_q(\Lambda) = \Sigma_{i \geqslant 2q(p-1)}\Lambda^i$. This filtration then induces filtrations $\{F_q\}$ on the Λ modules $BP_*(X)$ and $BP^*(X)$. The filtrations on Λ_p, $BP_*(X; \Lambda_p)$, and $BP^*(X; \Lambda_p)$ are obtained in a similar way. As before, we are using the symbol $\{F_q\}$ in an ambiguous manner.

If $H_*(X)$ is torsion free then the Atiyah-Hirzebruch-Whitehead spectral sequence converging from $H_*(X; \Lambda)$ to $BP_*(X)$ collapses (see 2.7 of **[12]**). This collapse implies that $BP_*(X)$ is a free Λ module. Thus $BP_*(X; G)$ is a free G module for all G. Also the homology maps ρ and T from 2.3 will be surjective. When we turn to $BP^*(X)$ the situation is more complicated because of the possible existence of "phantom" elements in $BP^*(X)$. These are nonzero elements of $BP^*(X)$ which map to zero in $BP^*(X^n)$ for all n (X^n is the nth skeleton). However, modulo these elements, our results for $BP_*(X)$ go through for $BP^*(X)$ as well (see 2.6 of **[12]** for the collapse of the AHW spectral sequence). In particular, modulo the phantom elements, $BP_*(X; G)$ and $BP^*(X; G)$ will be dual G modules for $G = \Lambda$ or Λ_p. Of course, the duality is in the sense of 2.1. Thus $BP_*(X; G)$ is connected and locally finite while $BP^*(X; G)$ need not be either. Finally, we might also add that all our dealings with $BP^*(X; G)$ could be stated in terms of $BP^*(X; G)$ mod F_n for some $n \geqslant 1$, and in such a situation all difficulties with $BP^*(X; G)$ disappear.

The algebra $BP^*(BP)$ of Brown-Peterson cohomology operations has been determined by Quillen in **[25]**. It is generated as a Λ module (provided we allow infinite sums) by elements $\{r_S\}$ where $S = (s_1, s_2, \ldots)$ ranges through all sequences of nonnegative integers with only finitely many nonzero terms. The element r_S is of dimension $|S| = \Sigma 2s_i(p^i - 1)$ (see **[26]** also for details).

When $BP^*(BP)$ and $BP_*(X)$ are interpreted as homotopy classes of maps between spectra then there is a natural left action of $BP^*(BP)$ on $BP_*(X)$. There is a natural left action of the Steenrod algebra $A^*(p)$ on $H_*(X; Z/p)$ obtained in the same way. It is related to the usual left action of $A^*(p)$ on $H^*(X; Z/p)$ via the rule

$$\langle\chi(\theta)x, y\rangle = (-1)^{|x||\theta|}\langle x, \theta y\rangle,$$

where $x \in H^*(X; Z/p)$, $y \in H_*(X; Z/p)$, $\theta \in A^*(p)$, and χ is the canonical antiautomorphism on $A^*(p)$ (see §3 of **[1]** for details on all of the above). For any sequence $S = (s_1, s_2, \ldots)$ of nonnegative integers with only finitely many nonzero terms, we can define the Milnor element $\mathcal{P}^S$ in $A^*(p)$ as in **[22]**. For any such S we have the relation

$$\begin{array}{ccc} BP_*(X) & \xrightarrow{r_S} & BP_*(X) \\ \downarrow \rho T & & \downarrow \rho T \\ H_*(X; Z/p) & \xrightarrow{\chi(\mathcal{P}^S)} & H_*(X; Z/p) \end{array} \tag{2.4}$$

This is another formulation of Lemma 3.7 of [**26**]. Zahler's result can be translated into 2.4 by using Proposition 2 from §3 of [**1**].

The algebra $BP^*(BP)$ acts on Λ and $H_*(BP)$. They are both polynomial algebras:

$$H_*(BP) = Q_p[m_1, m_2, \ldots] \quad \text{where } |m_s| = 2p^s - 2,$$

$$\Lambda = Q_p[v_1, v_2, \ldots] \quad \text{where } |v_s| = 2p^s - 2. \tag{2.5}$$

The Hurewicz homomorphism $h\colon \Lambda \to H_*(BP)$ is a monomorphism. Considering Λ as a subring of $H_*(BP)$ we have the identities

$$v_n = pm_n - \sum_{0<s<n} v_{n-s}^{p^s} m_s \tag{2.6}$$

(see [**13**]). In particular, h is an isomorphism when we tensor with Q. The morphism h respects the action of $BP^*(BP)$. The action of $BP^*(BP)$ on Λ and $H_*(BP)$ is nontrivial. For any sequence $R = (r_1, r_2, \ldots)$ of nonnegative integers with only finitely many nonzero terms, let $m(R) = m_1^{r_1} m_2^{r_2} \ldots m_r^{r_m}$. Let $|R| = \Sigma 2r_s(p^s - 1)$.

$$r_S(m(R)) = \begin{cases} 1 & \text{if } S = R, \\ 0 & \text{if } S \neq R \text{ and } |S| \geqslant |R|; \end{cases} \tag{2.7}$$

$$r_1(v_1) = p. \tag{2.8}$$

For 2.7 see [**26**]. For 2.8 we use 2.6 and 2.7.

3. A spectral sequence. In this section we discuss an Eilenberg-Moore type spectral sequence which relates the BP homology of ΩX to that of X. The spectral sequence arises from the fact that X has the same homotopy type as $B_{\Omega X}$, the classifying space of ΩX. The space $B_{\Omega X}$ is filtered by an increasing sequence $\{B_n\}$, where B_n is the n fold projective space of ΩX. This induces a filtration of $BP_*(B_{\Omega X}; G)$ and, hence, a spectral sequence $\{E_r\}$ such that:

$\{E_r\}$ is a first and fourth quadrant spectral sequence with differentials of bidegree $(-r, r-1)$; (3.1)

$E_2 = \mathrm{Tor}^{BP_*(\Omega X;\, G)}(G; G)$. In particular, the 1-stem (the elements with external degree 1) can be identified with $Q(BP_*(\Omega X; G))$; (3.2)

$$E_\infty = E^0(BP_*(X; G)). \tag{3.3}$$

The question of introducing an algebra structure into the spectral sequence is more involved. However, we will only use an algebra structure in a special case. Namely, E_2 will be an exterior algebra generated by the elements in the 1-stem. Then the spectral sequence collapses and we can identify the algebra structure of E_2 with that of E_∞.

The delooping map is related to this spectral sequence. To define the delooping map let $s: \Sigma\Omega X \to X$ be the adjoint to the identity map $\Omega X \to \Omega X$. It induces a map $\Omega_*: BP_*(\Omega X; G) \to BP_*(X; G)$ of degree $+1$. The map annihilates decomposables since it factors through $BP_*(\Sigma\Omega X; G)$. Further, it takes values in the primitives of $BP_*(X; G)$. To see this, observe that we have a commutative diagram

$$\begin{array}{ccc} BP_*(\Omega X; G) & \xrightarrow{(\Omega\Delta)_*} & BP_*(\Omega(X \times X); G) \\ \Omega_* \downarrow & & \downarrow \Omega_* \\ BP_*(X; G) & \xrightarrow{\Delta_*} & BP_*(X \times X; G) \end{array}$$

where Δ is the diagonal map. For any $x \in BP_*(\Omega X; G)$, $(\overline{\Omega\Delta})(x) = (\Omega\Delta)_*(x) - x \otimes 1 - 1 \otimes x$ is a finite sum of decomposables in $BP_*(\Omega(X \times X); G)$. Thus $(\overline{\Omega\Delta})(x)$ is annihilated by Ω_*. Hence Ω_* induces a map

$$\Omega_*: Q\big(BP_*(\Omega X; G)\big) \to P\big(BP_*(X; G)\big).$$

This delooping map can be reinterpreted in terms of the spectral sequence. We can identify $Q(BP_*(\Omega X; G))$ with the 1-stem of E_2 by 3.2. These elements are all permanent cycles in the spectral sequence. We can identify Image Ω_* with the image of these elements in E_∞.

As a reference for this spectral sequence see [**24**]. Further references are given there.

4. The structure of $H_*(\Omega X; \mathbf{Z}/p)$. In this section (X, μ) is a 1-connected H-space and $H_*(\Omega X)$ is torsion free. We will study the structure of $H_*(\Omega X; \mathbf{Z}/p)$ and $H^*(\Omega X; \mathbf{Z}/p)$ as Hopf algebras over the Steenrod algebra. In particular, they are bicommutative, biassociative Hopf algebras and dual to each other.

LEMMA 4.1. *If $x \in H_*(\Omega X; \mathbf{Z}/p)$ is of finite height then $x^p = 0$.*

For a proof of this lemma, see 4.2 of [**16**]. It then follows that

PROPOSITION 4.2. *$H_*(\Omega X; \mathbf{Z}/p)$ contains a sub-Hopf algebra T over the Steenrod algebra such that:*

(i) *$x^p = 0$ for all $x \in T$;*

(ii) *$H_*(\Omega X; \mathbf{Z}/p)$ is isomorphic, as an algebra, to $T \otimes P$, where P is a polynomial algebra.*

PROOF. Let $I = \{x \in H_*(\Omega X; \mathbf{Z}/p) | x^p = 0\}$. Then I is a Hopf ideal of

$H_*(\Omega X; \mathbf{Z}/p)$ over the Steenrod algebra. It is obviously an ideal. It is a Hopf ideal by formula (*) in (4.2) of **[16]**. It is invariant under the action of the Steenrod algebra by 3.4 of **[16]**. Let $P = H_*(\Omega X; \mathbf{Z}/p)/I$. Dualizing, P^* is a sub-Hopf algebra of $H^*(\Omega X; \mathbf{Z}/p)$ over the Steenrod algebra. Let $T^* = H^*(\Omega X; \mathbf{Z}/p)//P^*$. Finally, let $T = T^{**}$. Properties (i) and (ii) follow from 4.4 of **[22]**. Q.E.D.

For the rest of this section we will study T and its dual T^*. Since T is commutative, associative, and has only trivial pth powers, it follows that T^* is primitively generated (see 4.23 of **[23]**). Further, by the proof of 4.2, T^* is the quotient Hopf algebra $H^*(\Omega X; \mathbf{Z}/p)//P^*$ and inherits a Steenrod module structure from $H^*(\Omega X; \mathbf{Z}/p)$. We can use these facts to prove

LEMMA 4.3. *$Q^{2n}(T^*) = 0$ unless $n \equiv 1 \bmod p$.*

PROOF. We use the theory of secondary operations as outlined in §3 of **[18]**. The required secondary operations are defined on an element $x \in T^*$ by picking a representative of x from $H^*(\Omega X; \mathbf{Z}/p)$ and defining the secondary operation on this representative. The projection $H^*(\Omega X; \mathbf{Z}/p) \to T^*$ enables one to assume that the secondary operation takes values in T^*. Except in one case which we will mention, the fact that the Bockstein β_p acts trivially on $H^*(\Omega X; \mathbf{Z}/p)$ ensures that the required operations are always defined on the repesentatives in $H^*(\Omega X; \mathbf{Z}/p)$.

For p odd we use the arguments from **[18]**. Thus we use the secondary operations associated with the Adem relations

$$Q_0 \mathcal{P}^n = \sum (-1)^s \mathcal{P}^{n-\gamma(s)} Q_s$$

(Q_s are the Milnor elements from the Steenrod algebra (see **[22]**), while the function $\gamma(s)$ is defined by $\gamma(0) = 0$ and $\gamma(s) = \sum_{i=0}^{s-1} p^i$ for $s > 0$). Since T^* is primatively generated, the arguments from **[18]** simplify considerably in the present case.

For $p = 2$ we use arguments analogous to those in **[18]**. We first use the secondary operation ϕ_1 associated with the Adem relation

$$\mathrm{Sq}^1 \mathrm{Sq}^{4n} = \mathrm{Sq}^{4n} \mathrm{Sq}^1 + \mathrm{Sq}^2 \mathrm{Sq}^{4n-1}$$

$\left(a_1 = \mathrm{Sq}^{4n}, a_2 = \mathrm{Sq}^2, b_1 = \mathrm{Sq}^1, b_2 = \mathrm{Sq}^{4n-1}\right)$ to deduce that

$$Q^{4n}(T^*) = \mathrm{Sq}^2 Q^{4n-2}(T^*) \quad \text{for all } n \geqslant 1. \tag{*}$$

In particular, $Q^4(T^*) = 0$. We then use the operation ϕ_2 associated with the Adem relation

$$\mathrm{Sq}^1 \mathrm{Sq}^{4n} = \mathrm{Sq}^{4n} \mathrm{Sq}^1 + \mathrm{Sq}^2 \mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^{4(n-1)}$$

$\left(a_1 = \mathrm{Sq}^{4n}, a_2 = \mathrm{Sq}^2 \mathrm{Sq}^1, b_1 = \mathrm{Sq}^1, b_2 = \mathrm{Sq}^2 \mathrm{Sq}^{4(n-1)}\right)$ to deduce that

$$Q^{4n}(T^*) = 0 \quad \text{for } n \geqslant 1.$$

In this case, however, it is not automatic that ϕ_2 is defined. We must be sure that there exist representatives on which $b_2 = \mathrm{Sq}^2\mathrm{Sq}^{4(n-1)}$ acts trivially. Now any element of $Q^{4n}(T^*)$ has a primitive representative in $H^*(\Omega X; \mathbf{Z}/p)$. For, by the dual of 4.2(ii), $H^*(\Omega X; \mathbf{Z}/p)$ is isomorphic, as a coalgebra, to $T^* \otimes P^*$. But b_2 acts trivially on any primitive element $x \in H^{4n}(\Omega X; \mathbf{Z}/p)$. For, by 4.21 of [23], $b_2(x) \in H^{8n-2}(\Omega X; \mathbf{Z}/p)$ is either nondecomposable or a perfect square. But $b_2(x)$ cannot be nondecomposable because of $(*)$ and the relation $\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1$. And $b_2(x) = y^2$ is not possible since $y^2 = \mathrm{Sq}^1\mathrm{Sq}^{4n-2}(y)$. Q.E.D.

REMARK. An argument analogous to the above for $p = 2$ appears in **[20]** (see, in particular, 8.3 of **[20]**).

Now, since T^* is primitively generated, it is isomorphic, as a Hopf algebra, to a tensor product $T_e^* \otimes T_0^*$, where T_e^* is generated by primitive elements of even dimension and T_0^* is generated by primitive elements of odd dimension (see 7.16 of [23]). Write $T_e^* = \otimes_{s \in S} A_s$, where each A_s is generated by a single a_s of even dimension. For each a_s let h_s be defined by the rule that $p^{h_s} =$ the height of a_s in T^* ($h_s = \infty$ if height $= \infty$).

By dualizing 4.2(ii) we have that $H^*(\Omega X; \mathbf{Z}/p)$ is isomorphic, as a coalgebra, to $T^* \otimes P^*$. Under this (nonunique) isomorphism we can consider T^* as lying in $H^*(\Omega X; \mathbf{Z}/p)$. We will do so for the rest of this section.

For each a_s let $b_s = \mathcal{P}^{pn}(a_s)$ where $|a_s| = 2pn + 2$. Then $a_s^p = \mathcal{P}^1(b_s)$. Let

$A = \{a_s^{p^t} | s \in S, 0 \leqslant t < h_s\}$,

$A^+ = A \cup \{a_s^{p^{h_s}} | h_s \text{ is finite}\}$,

$B = \{b_s^{p^t} | s \in S, 0 \leqslant t < h_s\}$. The elements of A, A^+, B are related as follows.

LEMMA 4.4. (i) $\mathcal{P}^{p^{t+1}n}(a_s^{p^t}) = b_s^{p^t}$ *for* $t \geqslant 0$.
(ii) $\mathcal{P}^{p^t}(b_s^{p^t}) = a_s^{p^{t+1}}$ *for* $t \geqslant 0$.

PROOF. By induction on t using the Cartan formula. The case $t = 0$ is true by definition. Q.E.D.

LEMMA 4.5. A^+ *is a linearly independent set.*

PROOF. We need only show $A^+ - A$ is linearly independent since any relation of linear dependence involving elements from A would project to one in T^* via the quotient map $H^*(\Omega X; \mathbf{Z}/p) \to T^*$. So, suppose that we have a linear combination $x = \Sigma \lambda_s a_s^{p^{h_s}}$ among the elements of $A^+ - A$ with nontrivial coefficients. We wish to show that $x \neq 0$. Now $y = \Sigma \lambda_s a_s^{p^{h_s - 1}} \neq 0$ since A is linearly independent. By 3.11 of [7] y has 1-implication. By 4.2(i) and the remark which follows 3.11 of [7], the only possibility is that $x = y^p \neq 0$. Q.E.D.

LEMMA 4.6. B *is a linearly independent set.*

PROOF. This follows from 4.4(ii) and 4.5. Q.E.D.

The nonzero monomials in the elements $\{a_s\} \cup \{\text{generators of } T_0^*\}$ determine a basis of T^*. With respect to this basis let $x_s(t)$ be the element in T which is dual to $a_s^{p^t}$. Let

$\chi_1 = \{x_s(t) | s \in S, 0 \leq t < h_s\}$,

$\chi_2 =$ polynomial generators for P,

$\chi_3 =$ exterior algebra generators of T. Then 4.2(ii) can be rewritten as

PROPOSITION 4.7. *$H_*(\Omega X; \mathbf{Z}/p)$ is isomorphic, as an algebra, to $\mathbf{Z}/p[\chi_1]/I \otimes \mathbf{Z}/p[\chi_2] \otimes E[\chi_3]$, where I is the ideal generated by $\{x^p | x \in \chi_1\}$.*

The last few results are true for any choice of the imbedding $T^* \to H^*(\Omega X; \mathbf{Z}/p)$. We now make a particular choice of the imbedding so as to ensure that another property will hold. By taking all nonzero monomials in the elements of $\chi = \chi_1 \cup \chi_2 \cup \chi_3$ we have a basis of $H_*(\Omega X; \mathbf{Z}/p)$. For this basis we want to assume

LEMMA 4.8. *For $1 \leq n \leq p - 1$ the dual of $x_s(t)^n$ is $a_s^{p^t n}$ (up to a unit in $\mathbf{Z}/p$).*

PROOF. The lemma is true for the duality between T and T^* since $T^* = (\otimes A_s) \otimes T_0^*$ as a Hopf algebra. Thus, to show that the lemma is true for the duality between $H_*(\Omega X; \mathbf{Z}/p)$ and $H^*(\Omega X; \mathbf{Z}/p)$, we must show that, when T^* is considered as lying in $H^*(\Omega X; \mathbf{Z}/p)$, then

$$a_s^t \in T^* \quad \text{for } 0 \leq t < p^{h_s}. \tag{*}$$

So make a particular choice for the isomorphism 4.2(ii) and, hence, of the isomorphism $H^*(\Omega X; \mathbf{Z}/p) \cong T^* \otimes P^*$. This determines the elements $\{a_s\}$ in $H^*(\Omega X; \mathbf{Z}/p)$. Let $U = \{a_s^t | s \in S, 0 \leq t < p^{h_s}\}$. Pick a set of polynomial generators of $P \subset H_*(\Omega X; \mathbf{Z}/p)$. These generators are zero when evaluated on $\{a_s\}$, but they may be nonzero when evaluated on other elements of U. However, we can rewrite the polynomial generators (using elements of T) to ensure that they are zero when evaluated on U. Then, by induction on dimension, all elements in the ideal of $H_*(\Omega X; \mathbf{Z}/p)$ generated by these new generators are zero when evaluated on U. For, given $x, y \in H_*(\Omega X; \mathbf{Z}/p)$ and $a_s^t \in U$, we have the identities

$$\langle a_s^t, xy \rangle = \langle a_s^t, (\Omega\mu)_*(x \otimes y) \rangle = \langle (\Omega\mu)^*(a_s^t), x \otimes y \rangle.$$

Now, this new choice of polynomial generators for $H_*(\Omega X; \mathbf{Z}/p)$ amounts to a new choice of the isomorphism 4.2(ii) and, hence, of the isomorphism $H^*(\Omega X; \mathbf{Z}/p) \cong T^* \otimes P^*$. But the elements $\{a_s\}$ determined by this new isomorphism remain the same. For there is a duality between $Q(H_*(\Omega X; \mathbf{Z}/p))$ and $P(H^*(\Omega X; \mathbf{Z}/p))$ (see 3.10 of [23]), and the rewriting of the polynomial generators only involved elements which are zero when evaluated on $P(H^*(\Omega X; \mathbf{Z}/p))$. Thus (*) is now satisfied. Q.E.D.

5. The structure of $BP_*(\Omega X)$. In this section we use the results of §4 to study the Hopf algebra structure of $BP_*(\Omega X)$. Again (X, μ) is a 1-connected H-space and $H_*(\Omega X)$ is torsion free. We have the surjective maps

$$BP_*(\Omega X) \xrightarrow{\rho} BP_*(\Omega X; \Lambda_p) \xrightarrow{T} H_*(\Omega X; \mathbf{Z}/p).$$

In fact,

$$BP_*(\Omega X; \Lambda_p) = BP_*(\Omega X) \otimes \mathbf{Z}/p$$

and

$$H_*(\Omega X; \mathbf{Z}/p) = BP_*(\Omega X; \Lambda_p) \otimes_{\Lambda_p} \mathbf{Z}/p.$$

Let $\hat{\chi} = \hat{\chi}_1 \cup \hat{\chi}_2 \cup \hat{\chi}_3 = \{X_i\}$ be a set of representatives in $BP_*(\Omega X)$ for the elements $\chi = \chi_1 \cup \chi_2 \cup \chi_3 = \{x_i\}$ in $H_*(\Omega X; \mathbf{Z}/p)$. Here X_i denotes the representative of x_i. We will use $X_s(t)$ to denote the representative of $x_s(t)$. Let D be the set of nonzero monomials in the elements of $\hat{\chi}$ of weight $\geqslant 2$ which do not include the pth power of any element from $\hat{\chi}_1$. Thus $\hat{\chi} \cup D$ is a Λ basis of $BP_*(\Omega X)$. In fact it follows from 4.7 that

PROPOSITION 5.1. *$BP_*(\Omega X)$ is isomorphic, as an algebra, to $\Lambda[\hat{\chi}]/J$, where J is the ideal generated by $\{R_X | X \in \hat{\chi}_1\}$ and each R_X is of the form*

$$R_X = X^p - \sum \lambda_i X_i - \sum \omega_j d_j$$

for some $X_i \in \hat{\chi}$, $d_j \in D$ and λ_i, $\omega_j \in \Lambda$.

(Therefore, J defines the relations by which monomials in $\hat{\chi}$ involving pth powers of elements from $\hat{\chi}_1$ can be written in terms of $\hat{\chi} \cup D$.) Thus 5.1 demonstrates how the algebra structure of $BP_*(\Omega X)$ reflects the algebra structure of $H_*(\Omega X; \mathbf{Z}/p)$. We now show that the algebra structure of $BP_*(\Omega X)$ actually reflects the structure of $H_*(\Omega X; \mathbf{Z}/p)$ as a Hopf algebra over the Steenrod algebra.

For $n \geqslant 2$ define the map

$$(\Omega\Delta)_n: BP_*(\Omega X) \to \bigotimes_{i=1}^{n} BP_*(\Omega X)$$

by the recursive formula that

$$(\Omega\Delta)_2 = \text{the reduced multiplication } (\overline{\Omega\Delta})_*$$

defined in §3 and

$$(\Omega\Delta)_n = ((\Omega\Delta)_{n-1} \otimes 1)(\overline{\Omega\Delta})_* \quad \text{for } n > 2.$$

We can define $(\Omega\Delta)_n$ for the cases $H_*(\Omega X)$, $H_*(\Omega X; \mathbf{Z}/p)$ and $BP_*(\Omega X; \Lambda_p)$ in a similar manner. Proposition 5.1 holds for any choice of representatives $\hat{\chi}$.

If we make a more particular choice then we can assume

PROPOSITION 5.2. *Let* $X \in \hat{\chi}_1$ *such that* $|X| \equiv 2 \pmod{2p}$. *Then there exists* $Y \in BP_*(\Omega X)$ *such that*:

(i) $(\Omega\Delta)_p(Y) \equiv X \otimes \cdots \otimes X \pmod{F_1}$;

(ii) $R_X \equiv pY - v_1 r_1(Y) + v_1 d \pmod{F_2}$ *where* d *is decomposable.*

PROOF. In Part I we will produce Y, while in Parts II and III we will show that Y satisfies the required properties.

(I) $X = X_s(t)$ for some s and t. Thus X is an arbitrary representative (via the map $T\rho$) of $x_s(t) \in H_*(\Omega X; \mathbf{Z}/p)$ and, by diagram 2.3, we can determine X by first choosing $x \in H_*(\Omega X)$ such that $\rho(x) = x_s(t)$ and, then, choosing any $X \in BP_*(\Omega X)$ such that $T(X) = x$. But we claim that we can choose x such that it is primitive. First of all, $x_s(t)$ is primitive. For, since $|x_s(t)| \equiv 2$ mod $2p$, it follows that $t = 0$. Then the Hopf algebra decomposition $(\bigotimes A_s) \otimes T_0^*$ of T^* gives rise to a dual Hopf algebra decomposition of T showing that each $x_s(0)$ is primitive. Secondly, by 2.6 of [17], if $n \not\equiv 0$ mod $2p$ then $P(H_n(X; \mathbf{Z}/p)) = P(H_n(\Omega X)) \otimes \mathbf{Z}/p$. Thus $x_s(t)$ has a primitive representative x in $H_*(\Omega X)$.

Now $\rho(x^p) = x_s(t)^p = 0$. Define $y \in H_*(\Omega X)$ by $y = x^p/p$. Then y satisfies

(a) $(\Omega\Delta)_p(y) = x \otimes \cdots \otimes x$,

(b) $x^p = py$.

Finally, pick $Y \in BP_*(\Omega X)$ such that $T(Y) = y$.

(II) (i) follows from Part I(a).

(III) For (ii) we first observe that Part I(b) implies that

(c) $X^p \equiv pY + v_1 Z \pmod{F_2}$ for some $Z \in BP_*(\Omega X)$ (possibly zero).

Thus, in $Q(BP_*(\Omega X))$, $pY + v_1 Z \equiv 0 \pmod{F_2}$. Taking the image under the Quillen operation r_1 and using the fact that $r_1(v_1) = p$ and $r_1(F_2) \subset F_1$, it follows that, in $Q(BP_*(\Omega X))$,

(d) $pr_1(Y) + pZ \equiv 0 \pmod{F_1}$.

Taking the image of (d) under $Q(T)$: $Q(BP_*(\Omega X)) \to Q(H_*(\Omega X))$ and using the fact that $Q(H_{2n}(\Omega X))$ is torsion free if $n \equiv 1 \pmod{p}$ (see 2.2 of [17]), we deduce that, in $Q(H_*(\Omega X))$,

(e) $Tr_1(Y) + T(Z) = 0$.

Thus, in $Q(BP_*(\Omega X))$, $r_1(Y) + Z \equiv 0 \pmod{F_1}$. And, in $BP_*(\Omega X)$,

(f) $Z \equiv -r_1(Y) + d \pmod{F_1}$.

The required property of Y now follows by combining (c) and (f). Q.E.D.

REMARK. The relation of 5.2 to the Steenrod module structure of $H_*(\Omega X; \mathbf{Z}/p)$ arises from 2.4. This connection plays an important role in the proof of 6.2.

Pick representatives $\{A_s\}$ and $\{B_s\}$ in $BP^*(\Omega X)$ for the elements $\{a_s\}$ and

$\{b_s\}$ in $P(H^*(\Omega X; \mathbf{Z}/p))$. Moreover, pick the elements $\{B_s\}$ so that

$$\langle B_s, D\rangle = 0 \quad \text{for each } B_s. \tag{5.3}$$

These elements, as well as property 5.3, will be used in the next section when we study $BP_*(\Omega X; \Lambda_p)$.

6. The structure of $BP_*(\Omega X; \Lambda_p)$. This section is a continuation of §5. In this section we reduce mod p and study $BP_*(\Omega X; \Lambda_p) = BP_*(\Omega X) \otimes \mathbf{Z}/p$. Again (X, μ) is a 1-connected H-space and $H_*(\Omega X)$ is torsion free. We will use the same symbols to denote both elements in $BP_*(\Omega X)$ and their images in $BP_*(\Omega X; \Lambda_p)$. Thus all the elements defined in §5 can now be considered as lying in $BP_*(\Omega X; \Lambda_p)$. Further, the properties proved in §5 are still valid. In particular,

PROPOSITION 6.1. *$BP_*(\Omega X; \Lambda_p)$ is isomorphic, as an algebra, to $\Lambda[\hat{\chi}]/J$, where $\hat{\chi}$ and J are as in* 5.1.

The rest of this section will be devoted to proving a technical result about $BP_*(\Omega X; \Lambda_p)$. This will be a key result for the proof of Theorem 1.1. We will show

PROPOSITION 6.2. *If d is a monomial of weight $\geqslant 2$ in the elements of $\hat{\chi}$ and $\langle B_s^{p'}, d\rangle \not\equiv 0 \pmod{F_{p'+1}}$, then $d = X_s(t)^p$. Further,*

$$\left\langle B_s^{p'}, X_s(t)^p\right\rangle = v_1^p \quad (\textit{up to a unit in } \mathbf{Z}/p).$$

We first show that it suffices to prove 6.2 for the case $t = 0$. The map $\Omega\mu$: $\Omega(X \times X) \to \Omega X$ induces the reduced comultiplication map

$$\overline{\Omega\mu} : BP^*(\Omega X; \Lambda_p) \to BP^*(\Omega(X \times X); \Lambda_p)$$

defined as in §3. For any $n \geqslant 1$ we can identify $BP^*(\Omega(X \times X); \Lambda_p) \pmod{F_n}$ with $BP^*(\Omega X; \Lambda_p) \otimes BP^*(\Omega X; \Lambda_p) \pmod{F_n}$. If 6.2 is true for $t = 0$, then, by 4.8, we have, up to a unit in $\mathbf{Z}/p$, the identity

$$\overline{\Omega\mu}(B_s) \equiv v_1 \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} A_s^{p-i} \otimes A_s^i \pmod{F_2}.$$

But $\overline{\Omega\mu}(x^p) = [\overline{\Omega\mu}(x)]^p$ for any $x \in BP_*(\Omega X; \Lambda_p)$. Thus, for any $t \geqslant 0$,

$$\overline{\Omega\mu}\left(B_s^{p'}\right) \equiv v_1^{p'} \sum \frac{1}{p}\binom{p}{i} A_s^{p'(p-i)} \otimes A_s^{p'i} \pmod{F_{p'+1}}.$$

And, by 4.8, this is equivalent to 6.2.

Secondly, we prove 6.2 for the case $t = 0$. We will use

LEMMA 6.3. *If $d \in BP_*(\Omega X)$ is decomposable then $\langle B_s, d\rangle \equiv 0 \pmod{F_1}$.*

PROOF. We have the identities $T\langle B_s, d\rangle = \langle T(B_s), T(d)\rangle = 0$, the last inequality following from the fact that $b_s = T(B_s)$ is primitive while $T(d)$ is decomposable. Q.E.D.

We use this lemma to prove

LEMMA 6.4. *If d is a monomial of weight $\geqslant 2$ in the elments of $\hat{\chi}$ and $\langle B_s, d\rangle \not\equiv 0$ (mod F_2), then $d = X^p$ for some $X \in \hat{\chi}_1$.*

PROOF. (i) We can find $X \in \hat{\chi}_1$ such that X^p is a factor in d. For, otherwise, $d \in D$, which contradicts 5.3. Hence $d = X^pY$ for some Y.

(ii) We show Y is a unit in $\mathbf{Z}/p$. By 6.1, $X^p = \Sigma\lambda_i X_i + \Sigma\omega_j d_j$. Hence

$$d = X^pY = \sum \lambda_i X_i Y + \sum \omega_j d_j Y. \qquad (*)$$

If Y is not a unit then, by 6.3,

$$\langle B_s, X_i Y\rangle \equiv \langle B_s, d_j Y\rangle \equiv 0 \pmod{F_1}$$

for all i, j. Thus

$$\langle B_s, \lambda_i X_i Y\rangle = \lambda_i\langle B_s, X_i Y\rangle \equiv 0 \pmod{F_2}$$

and

$$\langle B_s, \omega_j d_j Y\rangle = \omega_j\langle B_s, d_j Y\rangle \equiv 0 \quad (\text{mod } F_2).$$

But (*) then implies that $\langle B_s, d\rangle \equiv 0$ (mod F_2), contradicting our hypothesis. Q.E.D.

We can now finish the proof of 6.2 for the case $t = 0$. By 6.4 we are left with eliminating the possibility that $d = X^p$, where $X \in \hat{\chi}_1$, but $X \neq X_s(0)$, and then showing that $\langle B_s, X_s(0)^p\rangle = v_1$. For dimension reasons we can eliminate all X except those having the same dimension as $X_s(0)$. In particular, $|X| \equiv 2$ (mod $2p$). Given such an X let Y be as in 5.2. Let $y = \rho T(Y)$ be its image in $H_*(\Omega X; \mathbf{Z}/p)$. We have the following sequence of (mod F_2) identities:

$$\begin{aligned}\langle B_s, X^p\rangle &\equiv \langle B_s, v_1 r_1(Y)\rangle + \langle B_s, v_1 d\rangle \quad (\text{by } 5.2)\\ &\equiv v_1\langle B_s, r_1(Y)\rangle \quad (\text{by } 6.3)\\ &\equiv v_1\langle T(B_s), Tr_1(Y)\rangle\\ &= v_1\langle b_s, \mathcal{P}^1(y)\rangle \quad (\text{by } 2.4)\\ &= v_1\langle \mathcal{P}^1(b_s), y\rangle\\ &= v_1\langle a_s^p, y\rangle \quad (\text{by } 4.4).\end{aligned}$$

But 5.2(i) implies that $(\Omega\Delta)_p(y) = x_s(0) \otimes \cdots \otimes x_s(0)$ in $\otimes_{i=1}^p H_*(\Omega X; \mathbf{Z}/p)$. Thus $\langle a_s^p, y\rangle \neq 0$ if, and only if, $X = X_s(0)$. This establishes 6.2.

7. Proof of Theorem 1.1. In this section we prove Theorem 1.1. We will divide our proof into four parts. In the first part we reduce the proof of 1.1 to that of a technical result. The last three parts then prove this result.

Part I. *Reduction of the proof.* We begin our reductions with

Reduction A. It suffices to show that $\mathrm{Tor} = \mathrm{Tor}^{BP_*(\Omega X;\ \Lambda(1))}(\Lambda(1);\ \Lambda(1))$ has no p torsion.

To verify Reduction A observe, first of all, that Tor having no p torsion implies that the spectral sequence from §3 converging from Tor to $BP_*(X;\ \Lambda(1))$ collapses. For it is well known (see, for example, §3 of [17]) that the Eilenberg-Moore spectral sequence converging from $\mathrm{Tor}^{H_*(\Omega X;\ Q)}(Q;\ Q)$ to $H_*(X;\ Q)$ collapses. Hence, by 2.2(c) and (f), the spectral sequence converging from $\mathrm{Tor} \otimes Q = \mathrm{Tor}^{BP_*(\Omega X;\Lambda_0(1))}(\Lambda_0(1);\ \Lambda_0(1))$ to $BP_*(X;\ \Lambda_0(1))$ collapses. But if Tor has no p torsion then there is an imbedding $\mathrm{Tor} \to \mathrm{Tor} \otimes Q$ and, hence, the spectral sequence converging from Tor to $BP_*(X;\ \Lambda(1))$ also collapses.

Secondly, it follows that $BP_*(X;\ \Lambda(1))$ (= Tor as a $\Lambda(1)$ module) has no p torsion. Hence, by 2.2(e), there is an imbedding $BP_*(X;\ \Lambda(1)) \to BP_*(X;\ \Lambda_0(1))$. Thus, $BP_*(X;\ \Lambda(1))$ is torsion free. For, by 2.2(c) and (f), $BP_*(X;\ \Lambda_0(1))$ is torsion free (indeed it is a free $\Lambda_0(1)$ module).

Before making further reductions in the proof of 1.1 we examine Tor. Observe that 5.1 holds when $BP_*(\Omega X)$ is replaced by $BP_*(\Omega X;\ \Lambda(1)) = BP_*(\Omega X) \otimes_\Lambda \Lambda(1)$. This implies

LEMMA 7.1. *For* $G = \Lambda$ *or* $\Lambda(1)$ $\mathrm{Tor}^{BP_*(\Omega X;G)}(G;\ G)$ *is isomorphic, as an algebra, to the homology of the complex* $K \otimes L \otimes M$ *where*

$$K = \bigotimes_{X \in \hat{X}_1} \Gamma(t(X)), \qquad t(X) \text{ has bidegree } (2, 2p|X|),$$

$$L = \bigotimes_{X \in \hat{X}_1 \cup \hat{X}_2} E(s(X)), \qquad s(X) \text{ has bidegree } (1, |X|),$$

$$M = \bigotimes_{X \in \hat{X}_3} \Gamma(s(X)), \qquad s(X) \text{ has bidegree } (1, |X|),$$

and the differential d acts by the rule

$$ds(X) = 0, \quad d\gamma_i(s(X)) = 0, \quad d\gamma_i(t(X)) = Q_X \gamma_{i-1}(s(X)),$$

where Q_X *is determined from* R_X *of* 5.1 *by the rule* $Q_X = \Sigma \lambda_i s(X_i)$.

For details as to how 7.1 follows from 5.1, at least in the analogous case of $MU_*(X)$, see §§3 and 8 of [24].

By 7.1, $K \otimes L$ and M are differential sub-Hopf algebras of $K \otimes L \otimes M$ and $H(K \otimes L \otimes M) = H(K \otimes L) \otimes H(M)$. Furthermore, $H(M) = M$ is torsion free. Thus to show Tor has no p torsion we need only show $H(K \otimes L)$ has no p torsion (when $G = \Lambda(1)$). For each $n > 1$ define sub-Hopf algebras K_n and L_n of K and L by the rule

$$K_n = \otimes\Gamma(t(X)), \qquad t(X) \text{ has internal degree } \leqslant n,$$
$$L_n = \otimes E(s(X)), \qquad s(X) \text{ has internal degree } \leqslant n.$$

Then, by 7.1, $K_n \otimes L_n$ is a differential sub-Hopf algebra of $K \otimes L$ and $H(K \otimes L) = \lim_{n\to\infty} H(K_n \otimes L_n)$. Thus to show $H(K \otimes L)$ is p torsion free it suffices to show $H(K_n \otimes L_n)$ is p torsion free for each n.

The point in looking at $K_n \otimes L_n$ is that we can now use a Bockstein spectral sequence to study p torsion. For there is no possibility of divisible elements occuring in $H(K_n \otimes L_n)$ to destroy the effectiveness of the Bockstein spectral sequence. As usual the spectral sequence $\{B_r\}_{r\geqslant 1}$ is induced from the exact couple

$$\begin{array}{ccc} H(K_n \otimes L_n) & \xrightarrow{\ \lambda\ } & H(K_n \otimes L_n) \\ & {}_{\tau}\nwarrow \quad \swarrow_{\rho} & \\ & H(K_n \otimes L_n \otimes \mathbf{Z}/p) & \end{array}$$

where λ is multiplication by p and ρ is reduction mod p. We now have our second reduction

Reduction B. It suffices to show that $B_1 = H(K_n \otimes L_n \otimes \mathbf{Z}/p)$ is an exterior algebra generated by elements with external degree 1.

For we wish to show that the spectral sequence collapses or, in other words that ρ is surjective. Now ρ is an algebra map and, by 7.1, the elements in $H(K_n \otimes L_n \otimes \mathbf{Z}/p)$ with externl degree 1 lie in the image of ρ. Thus if these elements generate $H(K_n \otimes L_n \otimes \mathbf{Z}/p)$ as an algebra then ρ is surjective.

For each element in $K_n \otimes L_n$ we will use the same symbol to denote its image in $K_n \otimes L_n \otimes \mathbf{Z}/p$. Then, our third reduction is

Reduction C. It suffices to show that the elements $\{Q_X\}$ of dimension $\leqslant n$ represent part of a $\Lambda_p(1)$ basis of $Q(L_n \otimes \mathbf{Z}/p)$.

For it would then follow that $K_n \otimes L_n \otimes \mathbf{Z}/p$ is isomorphic, as a differential algebra, to a tensor product $\otimes A_i$ where each A_i is of the form $A_i = E(a)$, $da = 0$, or $A_i = \Gamma(b) \otimes E(a)$, $d\gamma_i(b) = a\gamma_{i-1}(b)$. Since the second type has trivial homology it follows that $H(K_n \otimes L_n \otimes \mathbf{Z}/p) = \otimes H(A_i)$ is an exterior algebra of the required type.

We have now reached the technical result which we wish to prove. However, rather than working with $K_n \otimes L_n \otimes \mathbf{Z}/p$, we will assume for the rest of this section that

$$\hat{\chi}_3 = \phi \text{ and } \hat{\chi} = \hat{\chi}_1 \cup \hat{\chi}_2 \text{ is finite} \tag{7.2}$$

and work with $K \otimes L \otimes \mathbf{Z}/p$. The result to be proved now becomes

LEMMA 7.3. *$\{Q_X\}$ represents part of a $\Lambda_p(1)$ basis of $Q(L \otimes \mathbf{Z}/p)$.*

Our proof of 7.3 using 7.2 translates directly into a proof that the $\{Q_X\}$ of dimension $\leqslant n$ represent part of a basis of $Q(L_n \otimes \mathbf{Z}/p)$. However, the argument is more complicated to express. To see this consider the following result.

LEMMA 7.4. *For* $G = \Lambda_p$ *or* $\Lambda_p(1)$, $\mathrm{Tor}^{BP_*(\Omega X;\, G)}(G; G)$ *is isomorphic, as an algebra, to the homology of* $K \otimes L \otimes \mathbf{Z}/p$ *where* K *and* L *are as in* 7.1.

This follows from 6.1 and 7.2. (See §§3 and 8 of [**24**] for details.) If we did not assume 7.2 then we would have that

$$\mathrm{Tor}^{BP_*(\Omega X; G)}(G; G) = H(K \otimes L \otimes M \otimes \mathbf{Z}/p)$$

and $H(K_n \otimes L_n \otimes \mathbf{Z}/p)$ would only be a submodule of $\mathrm{Tor}^{BP_*(\Omega X;\, G)}(G; G)$. It is this fact which would complicate the exposition of our proof.

The next three parts are devoted to showing that $\{Q_X\}$ is part of a basis of $Q(L \otimes \mathbf{Z}/p)$. Our method is to produce a specific basis V of $Q(L \otimes \mathbf{Z}/p)$ and then modify it to include the elements $\{Q_X\}$. In Part II we produce the basis V. In Part III we prove some properties of V. In Part IV we show that V can be modified in the appropriate manner.

Part II. *The basis* V *of* $Q(L)$. We need only obtain a basis for $Q(L \otimes \mathbf{Z}/p)$ when $G = \Lambda_p$ since the case $G = \Lambda_p(1)$ is obtained by tensoring by $\Lambda_p(1)$ over Λ_p. Therefore, throughout Part II, we will work with Λ_p coefficients. Our reason for reducing to the case $G = \Lambda_p$ is that there exist surjective maps

$$Q(L \otimes \mathbf{Z}/p) \xrightarrow{\gamma} Q\big(BP_*(\Omega X; \Lambda_p)\big) \xrightarrow{Q(T)} Q\big(H_*(\Omega X; \mathbf{Z}/p)\big). \qquad (7.5)$$

The map γ is obtained from the fact that, by 3.2, $Q(BP_*(\Omega X; \Lambda_p))$ can be identified with the 1-stem of $\mathrm{Tor}^{BP_*(\Omega X;\Lambda_p)}(\Lambda_p; \Lambda_p)$ which in turn, by 7.4, can be identified with $Q(L \otimes \mathbf{Z}/p)/N$, where N is the Λ_p module generated by $\{Q_X\}$. (Here, as for the rest of this section, we do not distinguish between an element and its image in a quotient module.) Under the composite $Q(T) \circ \gamma$ the Λ_p basis $\{sX | X \in \hat{\chi}\}$ of $Q(L \otimes \mathbf{Z}/p)$ is mapped to a $\mathbf{Z}/p$ basis, χ, of $Q(H_*(\Omega X; \mathbf{Z}/p))$. Hence basis elements of $Q(L \otimes \mathbf{Z}/p)$ can be put into 1-1 correspondence with those of $Q(H_*(\Omega X; \mathbf{Z}/p))$. Further, if a set $V \subset Q(L \otimes \mathbf{Z}/p)$ projects, via $Q(T) \circ \gamma$, to a $\mathbf{Z}/p$ basis of $Q(H_*(\Omega X; \mathbf{Z}/p))$, then V must be a Λ_p basis of $Q(L \otimes \mathbf{Z}/p)$. We will now produce such a V.

We define a Λ_p linear map

$$s\colon BP_*(\Omega X; \Lambda_p) \to Q(L \otimes \mathbf{Z}/p)$$

by the rule that if $Y = \Sigma \alpha_i X_i + \Sigma \omega_j d_j$ is the expansion of Y in terms of the basis $\hat{\chi} \cup D$ (see §6), then $sY = \Sigma \alpha_i s X_i$. There exists a commutative diagram

$$\begin{array}{ccc} & Q(L \otimes \mathbf{Z}/p) & \\ {}^{s}\nearrow & & \searrow^{\gamma} \\ BP_*(\Omega X;\, \Lambda_p) & \xrightarrow{\quad q \quad} & Q(BP_*(\Omega X;\, \Lambda_p)) \end{array} \qquad (7.6)$$

where q is the standard quotient map. Thus to produce $V \subset Q(L \otimes \mathbf{Z}/p)$ with the desired projection property we need only produce a subset of $BP_*(\Omega X;\, \Lambda_p)$ which projects to a basis of $Q(H_*(\Omega X;\, \mathbf{Z}/p))$ via the map $Q(T)q$.

Expand B of 4.6 to a basis C of $P(H^*(\Omega X;\, \mathbf{Z}/p))$. Let $\hat{B}$ be a set of representatives in $BP^*(\Omega X;\, \Lambda_p)$ for B where, for $0 \leq t < h_s$, $b_s^{p^t}$ is represented by $B_s^{p^t}$ with B_s chosen to satisfy 5.3 (these restrictions on $\hat{B}$ will not be used until Part III). Expand $\hat{B}$ to a set of representatives $\hat{C}$ for C. Let $\hat{\psi}$ be a set of elements in $BP_*(\Omega X;\, \Lambda_p)$ which have a Kronecker pairing with the elements of $\hat{C}$. That is, $\hat{C} = \{C_i\}$ and $\hat{\psi} = \{Y_i\}$ are indexed by the same set I and, for any $i, j \in I$, we have the relation

$$\langle C_i, Y_j \rangle = \delta_{ij} \quad \text{(the Kronecker delta).}$$

Since $\hat{C}$ projects via T to a basis of $P(H^*(\Omega X;\, \mathbf{Z}/p))$, it follows from the duality between $P(H^*(\Omega X;\, \mathbf{Z}/p))$ and $Q(H_*(\Omega X;\, \mathbf{Z}/p))$ (see 3.10 of [23]) that $\hat{\psi}$ projects via $Q(T)q$ to a basis of $Q(H_*(\Omega X;\, \mathbf{Z}/p))$. Let V be the image of $\hat{\psi}$ in $Q(L \otimes \mathbf{Z}/p)$ under s.

Part III. *Properties of the basis* V. Again, we need only prove the properties for the case $G = \Lambda_p$ since they will still be valid when we pass to $\Lambda_p(1)$ coefficients. So, again, as in Part II, we will work with Λ_p coefficients. Let

$$Y_s(t) = \text{the element in } \hat{\psi} \text{ which is dual to } B_s^{p^t},$$

$$Z_s(t) = \text{the image of } Y_s(t) \text{ in } V,$$

$U = \{Z_s(t)\}$, and $U' = \{Q_X\}$. When we pass to $\Lambda_p(1)$ coefficients in Part IV we will modify V by replacing the subset U by U'. So we wish to know something of the relation between U and U.

First of all, there is a natural 1-1 correspondence between the elements of U and U'. We deduce this by establishing, in turn, 1-1 correspondences between A and B (using 4.4), $\hat{A}$ and $\hat{B}$, $\hat{\chi}_1$ and $\hat{\psi}$, and finally U and U'. Next reindex the elements of U' as $Q_1, Q_2, \ldots, Q_k$, where $|Q_1| \leq |Q_2| \leq \cdots \leq |Q_k|$. Reindex the elements of U as $Z_1, Z_2, \ldots, Z_k$ by the convention that Z_n corresponds to Q_n. Then, when we expand Q_n in terms of the basis V, we have the following result.

LEMMA 7.7. *The coefficient of Z_n in Q_n is a power of v_1, while for $i > n$ the coefficient of Z_i in Q_n is trivial.*

We prove 7.7 by reducing to a result in $BP_*(\Omega X; \Lambda_p)$. First, $\hat{\psi} \cup D$ is a basis of $BP_*(\Omega X; \Lambda_p)$ since $\hat{\psi}$ projects to a basis of $Q(H_*(\Omega X; \mathbf{Z}/p))$, while D projects to a basis of the decomposable elements in $H_*(\Omega X; \mathbf{Z}/p)$. Secondly, the map s: $BP_*(\Omega X; \Lambda_p) \to Q(L \otimes \mathbf{Z}/p)$ can be redefined as follows. If $Y = \Sigma \alpha_i Y_i + \Sigma \omega_j d_j$ is the expansion of Y in terms of $\hat{\psi} \cup D$, then $s(Y) = \Sigma \alpha_i Z_i$. Thirdly, by the definition of Q_X, we have that $s(X^p) = Q_X$ for each $X \in \hat{\chi}_1$. Thus, to prove 7.7, it will suffice to show that when we expand X^p in terms of $\hat{\psi} \cup D$ then

LEMMA 7.8. *The coefficient of* $Y_s(t)$ *in* X^p *is* $v_1^{p'}$ *if* $X = X_s(t)$ *and trivial if* $X \neq X_s(t)$ *and* $|X^p| \leqslant |X_s(t)^p|$.

This follows from 6.2. Take the basis $\hat{\psi} \cup D$ of $BP_*(\Omega X; \Lambda_p)$ and pick a set of elements in $BP^*(\Omega X; \Lambda_p)$ which have a Kronecker pairing with the set $\hat{\psi} \cup D$. Pick a given s and t. Now $B_s^{p'}$ cannot necessarily be chosen as the element dual to $Y_s(t)$ since $B_s^{p'}$ may be nonzero when evaluated on the elements of D. However, 6.2 ensures that this does not happen in dimensions $\leqslant |X_s(t)^p|$. Furthermore, we can restrict our attention to these dimensions when proving 7.8 for the given s and t. Thus we can assume that $B_s^{p'}$ is the element chosen as the dual to $Y_s(t)$. Then, by dualizing 6.2, we obtain 7.8.

Part IV. *Modification of the basis* V. We now pass to $\Lambda_p(1)$ coefficients and work with these coefficients throughout Part IV. We wish to show that the set V' obtained through replacing U by U' is a $\Lambda_p(1)$ basis of $Q(L \otimes \mathbf{Z}/p)$. To do this we need only show that the elements of V can be written in terms of the elements from V'. Thus, without loss of generality, we can assume that $U = V$ and $U' = V'$. And then, by 7.7, when we form the $k \times k$ matrix $A = (a_{ij})$, where a_{ij} is the coefficient of Z_i in Q_j, our matrix will be of the form

$$A = \begin{vmatrix} v_1^{s_1} & * & * & \cdots & * \\ & v_1^{s_2} & * & \cdots & * \\ & & \ddots & & \vdots \\ & O & & \ddots & \vdots \\ & & & & v_1^{s_k} \end{vmatrix}$$

Since v_1 is invertible in $\Lambda_p(1)$, A is invertible, and hence U' can be written in terms of U.

8. Proofs of the remaining theorems. In this section we prove Theorems 1.2–1.4.

I. PROOF OF THEOREM 1.2. The first part of 1.2 is deduced as follows. By 2.2(c) and (f), $H_*(X; Q)$ is commutative (associative) if, and only if, $BP_*(X;$

$\Lambda_0(1))$ is commutative (associative). By 2.2(e) and the fact that $BP_*(X; \Lambda(1))$ is torsion free, $BP_*(X; \Lambda_0(1))$ is commutative (associative) if, and only if $BP_*(X; \Lambda(1))$ is commutative (associative).

For the second part of 1.2 observe that by the argument in §7 (in particular, see Reduction A) we have the identities

$$E^0\big(BP_*(X; \Lambda(1))\big) = E_\infty = E_2 = \mathrm{Tor}^{BP_*(X;\Lambda(1))}(\Lambda(1); \Lambda(1)).$$

Then, by the definition of Ω_* given in §3 it suffices to show that $\mathrm{Tor}^{BP_*(\Omega X;\Lambda(1))}(\Lambda(1); \Lambda(1))$ is generated by the elements with external degree 1. But this follows from a minor variation of the argument given in §7. For Lemma 7.3 implies that $H(K_n \otimes L_n \otimes \mathbf{Z}/p)$ is an exterior algebra generated by the elements with external degree 1. Thus the Bockstein spectral sequence collapses and $H(K_n \otimes L_n)$ is generated by the elements with external degree 1. But

$$\mathrm{Tor}^{BP_*(\Omega X;\Lambda(1))}(\Lambda(1); \Lambda(1)) = \lim_{n\to\infty} H(K_n \otimes L_n).$$

This follows from 7.1 plus the fact (see 2.2 and 3.1 of [17]) that $H^*(X; Q)$ an exterior algebra implies that $\chi_3 = \phi$. Thus $\mathrm{Tor}^{BP_*(\Omega X;\Lambda(1))}(\Lambda(1); \Lambda(1))$ is generated by the elements with external degree 1.

II. Proof of Theorem 1.3. The canonical inclusion $\Lambda \to \Omega$ enables us to define $MU_*(X; \Omega(1))$ by the rule

$$MU_*(X; \Omega(1)) = BP_*(X; \Lambda(1)) \otimes_{\Lambda(1)} \Omega(1).$$

Theorem 1.3 now follows from Theorems 1.1 and 1.2.

III. Proof of Theorem 1.4. We can pick polynomial generators of $\Omega = Q_p[t_1, t_2, \ldots]$ such that the Todd genus $T_d\colon \Omega \to Q_p$ is defined by $T_d(t_s) = 1$ for all s. Further, the nondecomposable v_1 is one of these generators (v_1 is the cobordism class of the complex manifold CP^{p-1}). The Conner-Floyd isomorphism asserts that $K_*(X) \otimes Q_p = (MU_*(X) \otimes Q_p) \otimes_\Omega Q_p$, where Q_p is considered as a Ω module via the Todd genus. The Todd genus can be extended to $Td'\colon \Omega(1) \to Q_p$ by the rule that $Td'(1/v_1) = 1$. Then the Conner-Floyd isomorphism can be rewritten as $K_*(X) \otimes Q_p = MU_*(X; \Omega(1)) \otimes_{\Omega(1)} Q_p$, and 1.4 follows from 1.3.

This proof of 1.4 also explains why we only localized with respect to v_1 when proving 1.1. For, in view of the rather complicated arguments given in §7, it might be thought that a simpler proof could be obtained if we localized with respect to more v_s for $s > 1$. However, the Todd genus has the property that $Td(V_s) = 0$ for $s > 1$. Hence, any results obtained by localizing with respect to v_s for any $s > 1$ cannot be pushed down into K-theory. And the K-theory results are our main interest.

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